Physics Across Oceanography: Fluid Mechanics and Waves

Physics Across Oceanography: Fluid Mechanics and Waves

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Contents

This is a primer on vector fields and key differential operators that are useful in fluid mechanics. It is currently under development for Ocean 285 *Physics Across Oceanography*, offered at the University of Washington.

PART I

VECTOR FIELD DIFFERENTIAL CALCULUS PRIMER

1. Coordinates

To mathematically describe variables in the ocean, we need three dimensions of space and one dimension in time. With more than one spatial dimension, position becomes a vector of distances along each dimension from some origin (0,0,0).

Often (and almost always in this class) it is convenient to use a local Cartesian coordinate system for a plane tangent to the Earth's surface. We write:

$$\underline{\mathbf{x}} = (x, y, z)$$

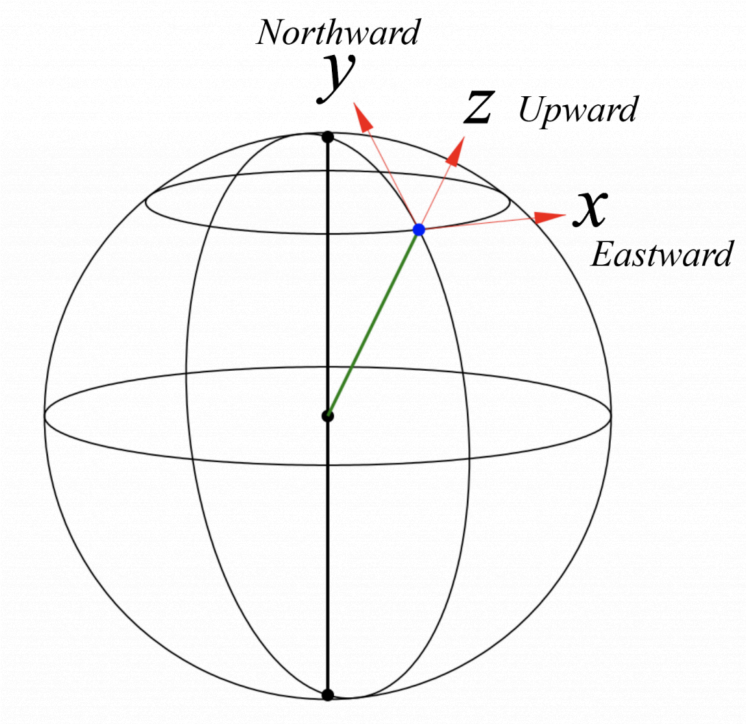
Key Takeaways

x increases to the East along the local horizontal plane, in the direction of increasing longitude

y increases to the North along the local horizontal plane, in the direction of increasing latitude

z increases upward in the local vertical direction

$z = 0$ is at the surface of the ocean (to be precise: where the ocean surface would be if there were no motion, i.e., the surface of a resting ocean in equilibrium with Earth's gravity aka geopotential field).



The local Cartesian coordinates for a point on the surface of the Earth at the blue dot.

Note 1: The absolute orientation of this standard local Cartesian coordinate system changes depending on where you are on the Earth (i.e., the direction of “Up” differs if seen from outer space).

Note 2: Sometimes, especially for coastal or estuarine problems, a different local Cartesian coordinate system is used. For example, x might point along the coastline, with y pointing offshore.

FYI: Latitude and longitude come from a spherical coordinate system. Longitude is the angle counterclockwise looking down on the north pole relative to 0° passing through Greenwich, England. Latitude is the angle counterclockwise looking at the Earth from above the equator (0° corresponds to the equator). Elevation (formally: distance from the center of the Earth) is the third coordinate in this system.

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2. Variables

A **field** in physics is a function that has a value at every point in space and time. One example is the gravitational field of the Earth. All of the variables we will use in this class are fields.

Using our convenient local Cartesian coordinate system, a **scalar** field has only a magnitude. Its value is a single number at any point in space, and time.

Examples

Some scalar fields in oceanography (scalar functions of the coordinates):

- Salinity, written mathematically as $S(x,y,z,t)$
- Pressure, $P(x,y,z,t)$
- Concentration of a dissolved substance, $C(x,y,z,t)$

A **vector** field has a three-dimensional vector at every point in space and time. Imagine space filled with small arrows with a direction and magnitude (length) that vary with spatial location, and also change in time.

Examples

Some vector fields in oceanography (these will be defined later):

- The fluid velocity, written as $\underline{\mathbf{u}}(x,y,z,t)$ has components (u, v, w) in each coordinate direction
- The flux of substance C, written as $\underline{\mathbf{F}}_C(x,y,z,t)$
- The pressure gradient force, written as $\underline{\mathbf{F}}_P(x,y,z,t)$

Key Takeaways

Let's define flux and transport. These are two important general kinds of vector fields in oceanography. We'll have more to say about these later.

- The **flux** of a property is the rate at which that property passes through some surface, per unit area of the surface.
- The **transport** of a property is its flux times the area of the surface.

Unit vectors have length = 1. You can create a unit vector in any direction by dividing the vector by its magnitude. In physics, this means that you have divided out the units and so the vector is *dimensionless*.

Special symbols are often given to unit vectors in the directions of the coordinate system axes. In our (x,y,z) = (Eastward, Northward, Upward) coordinate system:

$$\hat{i} = (1, 0, 0)$$
$$\hat{j} = (0, 1, 0)$$
$$\hat{k} = (0, 0, 1)$$

<u>Review Material</u> (hopefully these are blasts from your past):

- While a scalar has only a magnitude, a vector has both magnitude and direction.
- Vertical bars surrounding a vector indicate its magnitude.
- To calculate the magnitude of a vector: sum up the squares of the components, then take the square root of the total. Mathematically:

$$|\underline{\mathbf{u}}| = |(u, v, w)| = \sqrt{u^2 + v^2 + w^2} = \left(u^2 + v^2 + w^2\right)^{1/2}$$

3. The Dot Product

The first vector operator we will define is the **dot product.** For two vectors $\underline{\mathbf{b}} = (b^x, b^y, b^z)$, and $\underline{\mathbf{c}} = (c^x, c^y, c^z)$:

$$\underline{\mathbf{b}} \cdot \underline{\mathbf{c}} = b^x c^x + b^y c^y + b^z c^z$$

Geometrically the dot product gives the magnitude of the component of $\underline{\mathbf{b}}$ that is aligned with $\underline{\mathbf{c}}$, multiplied by the magnitude of $\underline{\mathbf{c}}$.

- If two vectors are perpendicular to one another, then the dot product is zero.
- If two vectors are parallel, then the dot product is $|\underline{\mathbf{b}}|$ times $|\underline{\mathbf{c}}|$.

Examples

The dot product of the fluid velocity and one of the Cartesian coordinate unit vectors gives the current component in that direction. For example, let's suppose that the current is 2.88 ms^{-1} to the northwest and upwelling at 0.01 cm s^{-1}

$$\underline{\mathbf{u}} \cdot \hat{j} = (-2, 2, 0.0001)ms^{-1} \cdot (0, 1, 0)$$

$$= -2 \times 0 + 2 \times 1 + 0.0001 \times 0 = 2ms^{-1}$$

gives the northward component of the fluid velocity.

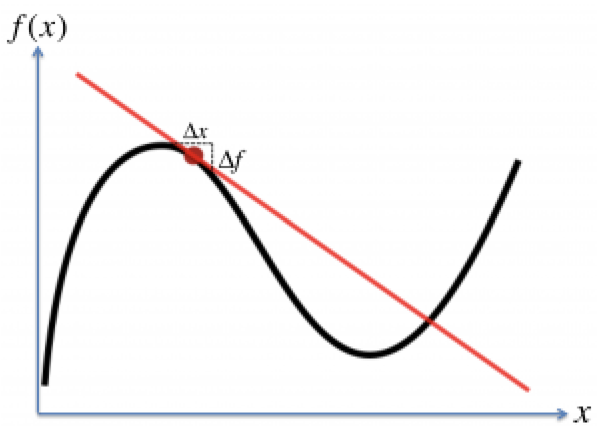
There is another important vector operator called the cross product, but we will define that a bit later.

4. Partial Derivatives

For a function of one *independent variable*, say x, the derivative gives information about how the function, say $f(x)$, changes when x changes. This is the meaning of the first derivative.

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x}$$

In other words, if you make a small but finite change Δx, you get a change in the value of the function Δf. The derivative is the ratio of these two changes, taken in the limit when the changes become very small. A geometrical interpretation is that the derivative is the slope of a tangent line. The derivative can also be a function of x (i.e., in the picture below, the slope of the tangent line will change as you move in x).



When we have something that is a function of **multiple** independent variables (like x, y, z and t), we define an equivalent operation that tells us how the function changes when just one of the independent variables changes and all the others are kept constant. This is called the *partial derivative*. It is written with a special symbol "∂" instead of the "d".

The partial derivative of f with respect to x is written

$$\frac{\partial f}{\partial x}$$

and has the meaning of a change in f divided by a change in x calculated while holding all the other independent variables constant.

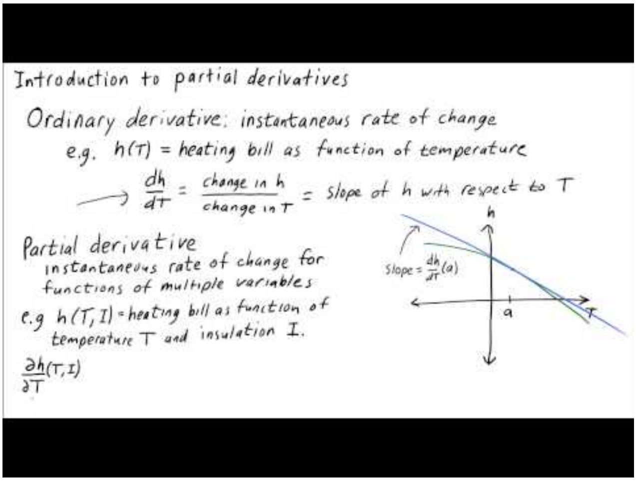
Key Takeaways

In fluid mechanics, we often write the operation of taking the derivative in each of the spatial dimensions as a vector operator:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)$$

While this may seem a bit abstract, the next two sections will be about two applications of this vector operator, in order to calculate the *gradient* and the *divergence*. These quantities will be fundamental to what we do in the rest of the course.

If you want more material to help you understand partial derivatives, try this YouTube video:



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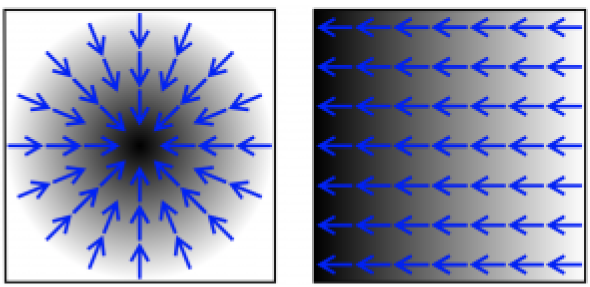
5. The Gradient

The **gradient** is a vector with a magnitude that quantifies the total amount of change of a field per unit change in distance. It points along the direction of the maximum change. The gradient is only defined for a *scalar* field. The operation of taking the gradient of a scalar produces a *vector*.

The easiest way to visualize this is with a two-dimensional gradient of concentration (a scalar) in a horizontal plane as in the figure below. In this figure, higher concentrations are shaded more darkly and the blue arrows show the concentration gradient vector field.



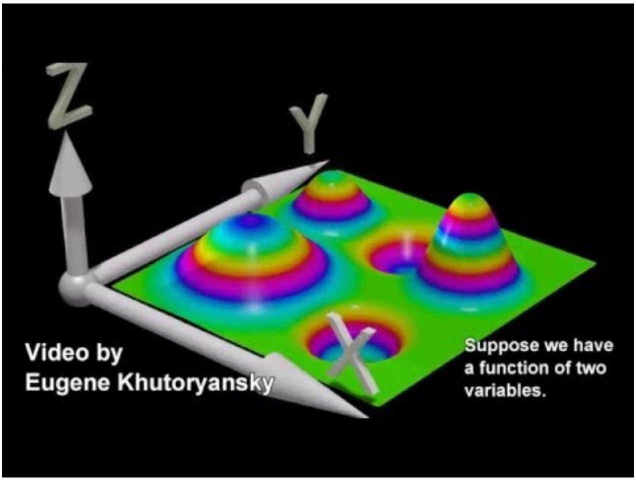
Key Takeaways

The mathematical definition of the gradient of the scalar field C (in our three standard spatial dimensions) is:

$$\nabla C = \left(\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}, \frac{\partial C}{\partial x}\right)$$

where we have used the partial derivative operator defined in the previous chapter.

For a highly visual discussion of partial derivatives and the gradient, see this YouTube video:



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6. The Divergence

The divergence of a vector is the sum of the partial derivatives of its components with respect to their individual coordinates.

Key Takeaways

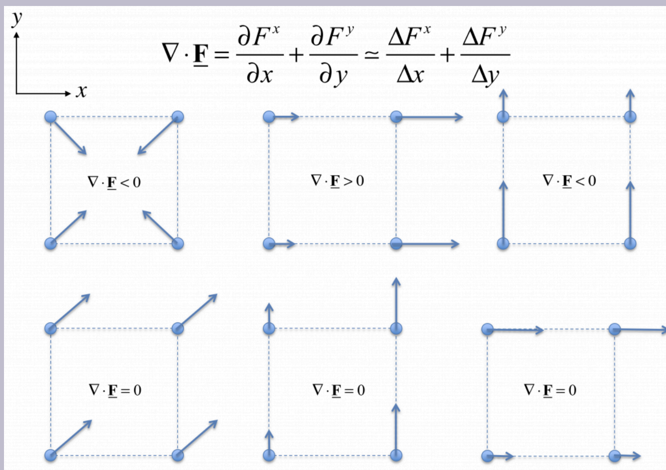
The divergence of a vector $\underline{\mathbf{F}}$, written in terms of its components in our usual coordinate directions $\underline{\mathbf{F}} = \left(F^x, F^y, F^z\right)$ is defined as:

$$\nabla \cdot \underline{\mathbf{F}} = \frac{\partial F^x}{\partial x} + \frac{\partial F^y}{\partial y} + \frac{\partial F^z}{\partial z}$$

You can get an intuitive feel for the divergence by sketching vectors in space and thinking about how these vectors are tending to stretch out or squash the distance between.

Examples

Here are some examples of vector fields, and the sign of their divergence in a horizontal plane:



We will be considering vectors that represent fluid velocity and the flux of some property like salinity, or oxygen, in the ocean. It is important to note that in the first case, where the vector field is velocity, the divergence of an "incompressible" fluid in three-dimensional space is zero because material elements of water cannot be stretched or squashed.

Key Takeaways

The divergence of the velocity field in an incompressible fluid is zero:

$$\nabla \cdot \underline{\mathbf{u}} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

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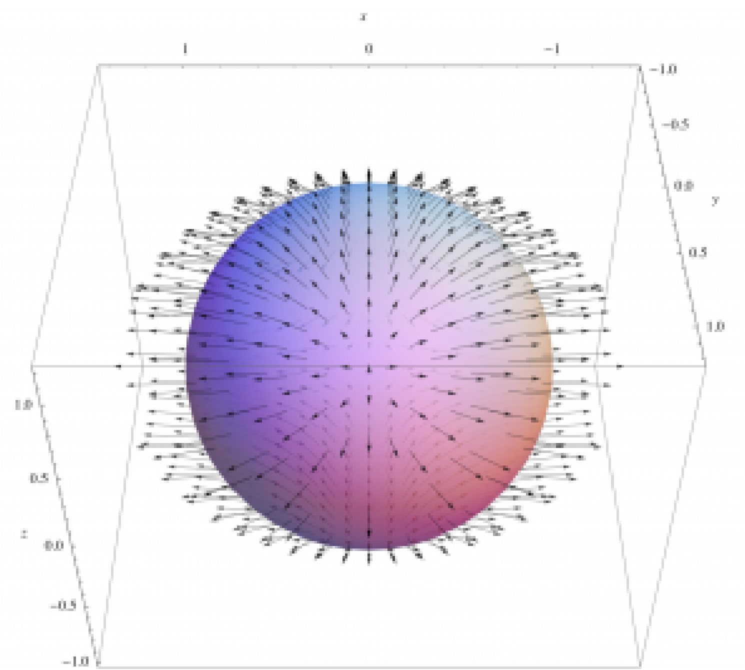
7. The Divergence Theorem

The divergence theorem is an equality that can be proven for any vector field. All of our conservation equations and large-scale budgets that are so useful in oceanography are based on this mathematical principle. The divergence theorem can be abstracted to any vector field, but is most easily understood by thinking about the concrete example where the vector field is the flux of some property in the ocean.

The divergence theorem relates the flux outward through the surrounding surface of a closed volume to the total divergence inside. Consider the figure below showing vector field $\underline{\mathbf{F}}$ and the surface of a sphere.



This theorem is usually stated mathematically using surface and volume integrals, but that is beyond the level of calculus expected for this class. Instead, we will approach it by imagining that the surface of the sphere is divided up into a very large number of surface elements.

The i-th surface area element:

- has surface area a_i.
- is associated with a local value of the vector field $\underline{\mathbf{F}}_i$
- has a unit *normal vector* $\hat{n}_i$ that is directed outward and perpendicular to this surface element
- has an outward flux passing through the surface $\underline{\mathbf{F}}_i \cdot \hat{n}_i$

> The total outward transport (flux times area) passing through the surface of the sphere = $\sum_i \left(\underline{\mathbf{F}}_i \cdot \hat{n}_i\right) a_i$

Next, we apply a similar idea to the volume inside the sphere, dividing it up into a very large number of volume elements.

The j-th interior volume element:

- has volume V_j
- is associated with a local vector $\underline{\mathbf{F}}_j$
- is associated with a local divergence of that vector field $\nabla \cdot \underline{\mathbf{F}}_j$

> The total divergence inside the sphere = $\sum_j \left(\nabla \cdot \underline{\mathbf{F}}_j\right) V_j$

Key Takeaways

The divergence theorem sets the net transport outward across the surface of a closed volume (e.g., the sphere in the figure above) equal to the total divergence inside the sphere.

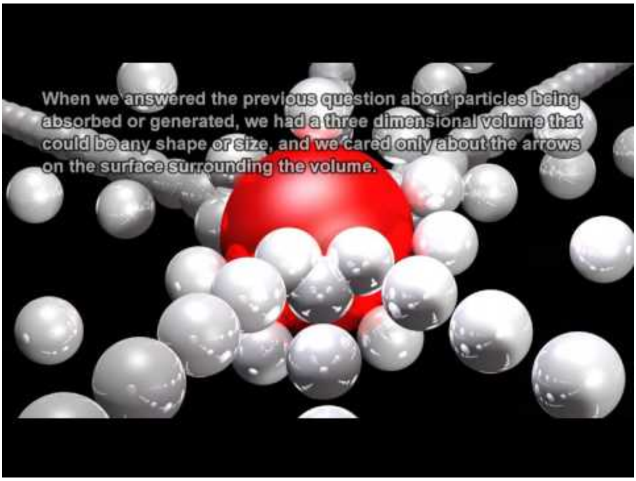
$$\sum_i \left(\underline{\mathbf{F}}_i \cdot \hat{n}_i\right) a_i = \sum_j \left(\nabla \cdot \underline{\mathbf{F}}_j\right) V_j$$

Practically, if there is a net outward transport of a property across some volume, there must be a source of that property inside the volume (in order for there to be a positive total divergence).

With this formal basis, we can relate measurements of fluxes across the sides of some "box" in the ocean to what happens inside, on average, and thereby develop budgets for various properties of interest. As an example, imagine taking flux measurements in Admiralty Inlet (through which most of the water passes into or out of Puget Sound), in the form of a vertical cross-section that reaches from the ocean surface to the bottom and that crosses from one side of the passage to the other. By measuring the total flux of oxygen across this surface, we can infer the net production (or uptake from the atmosphere) of oxygen in Puget Sound. If there is more oxygen leaving with the surface outflow than entering in the deep inflow, we know that oxygen is being added to the water inside.

We can also take our budget principles down to very small (aka "infinitesimal" volumes) and develop differential forms of the budget equations.

For a highly visual discussion of divergence and the divergence theorem, see the first half of this YouTube video:



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8. The Curl, and Vorticity

The third of our important partial differential operations is taking the curl of a vector field. This produces another vector.

Key Takeaways

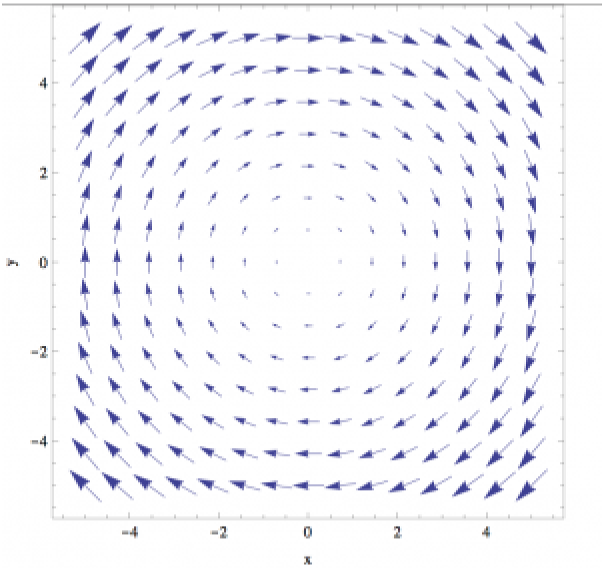
The curl of the vector field $\underline{\mathbf{F}} = (F^x, F^y, F^z)$ is defined as:

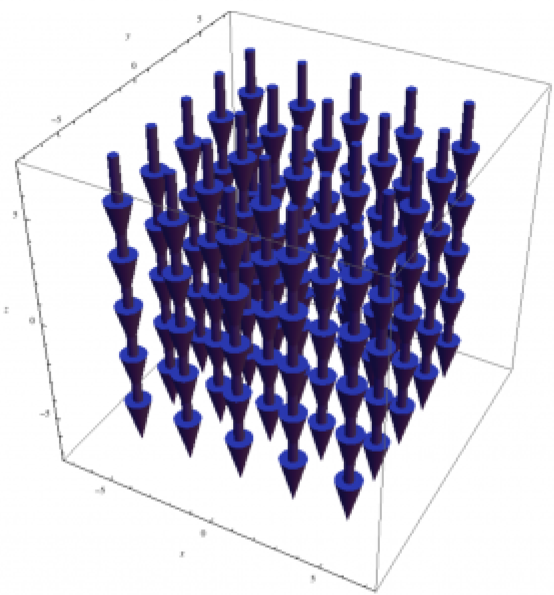
$$\nabla \times \underline{\mathbf{F}} = \left(\frac{\partial F^z}{\partial y} - \frac{\partial F^y}{\partial z}, \frac{\partial F^x}{\partial z} - \frac{\partial F^z}{\partial x}, \frac{\partial F^y}{\partial x} - \frac{\partial F^x}{\partial y} \right)$$

We are only going to be concerned with the curl of a two-dimensional vector field in the horizontal plane in this class. One important example is the curl of the horizontal velocity which is the definition of **vorticity**, commonly written as ζ or ω (we will use ζ). Since the z-component is zero in this case, we see that the vorticity only has a vertical component.

$$\zeta = \nabla \times \underline{\mathbf{u}}_{HORIZ} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Physically, the vorticity is a metric of the local spinning motion of a fluid, in this case around the vertical axis. The sign of the vorticity relative to “Up” follows the right-hand rule: if you wrap your fingers in the direction of the circulation of the vectors in the horizontal plane, your thumb will point in the direction of the curl vector in the vertical plane.

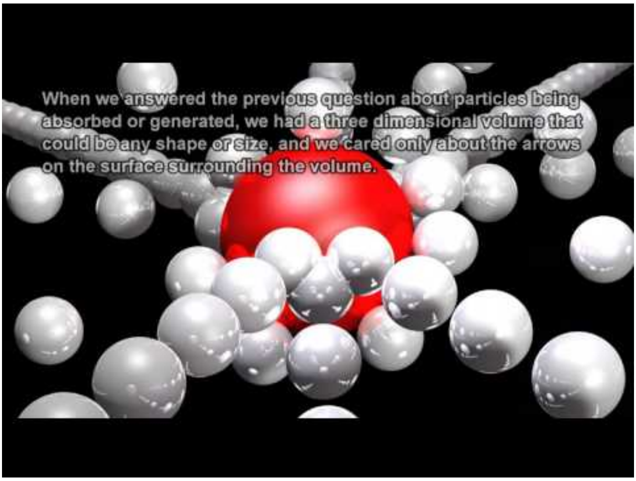




As an example, imagine the North Pacific subtropical gyre, a clockwise circulation. To curl your fingers in a clockwise manner (in the horizontal plane) with your right hand, your thumb must point downward. Thus the large-scale curl of velocity for the subtropical gyre is negative.

There is a fundamental theorem for curl that is analogous to the Divergence Theorem that you will learn about if you take Multivariate Calculus. It is called the Circulation Theorem. But we won't go there in this class.

For a highly visual discussion of curl see the second half of this YouTube video (you watched the first half in the last chapter):



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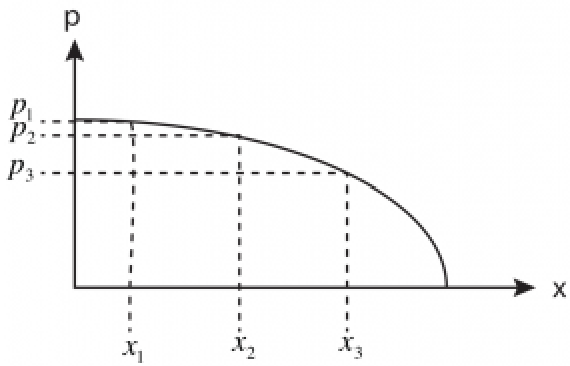
9. Estimating Derivatives from Data

If we want to use these ideas in real world problems, we probably need to be able to estimate partial derivatives using data.

Here is some pretend data – for concreteness' sake let's say that the y-axis is pressure (perhaps from from sea-surface height measured by a satellite altimeter). For example, it might be the pressure as a function of distance to the east (x) holding depth (the z-coordinate) and latitude (the y-coordinate) fixed.



$p(x)$ at z = -50 m and y = 24°N

We want to use our measurements of pressure at discrete values of longitude (x_i) to estimate the first- and second- partial derivatives of pressure with x. For example, we could use the first partial derivative to estimate the pressure gradient at, say, x_2. As another example, when we work problems involving diffusion, we might need to estimate the second partial derivatives.

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9.1 Estimating the first partial derivative

In the example above, the first partial derivative at x_2 is approximated by the change in p divided by the change in x (i.e., the "rise" over the "run" to get the approximate slope of the tangent line at x_2.

Key Takeaways

This is called a finite difference approximation to the partial derivative of pressure with respect to x:

$$\left.\frac{\partial p}{\partial x}\right|_{x_2} \approx \frac{\Delta p}{\Delta x} = \frac{p(x_3) - p(x_1)}{x_3 - x_1}$$

Why did we use points 1 and 3? If we used either points 1 and 2, then the resulting tangent line would have a slope that is more appropriate to a point that lies *between* points 1 and 2. Since we want the tangent line slope at x_2, we choose to take a difference using points that are centered on point 2. Not surprisingly, this is called using a centered difference approximation. Had we used points 1 and 2 to calculate our Δ values, then we would have used a backward difference approximation (because we used information for values lower than, or behind, x_2). Had we used points 2 and 3, we would have used a forward difference approximation.

9.2 Estimating the second partial derivative

The second derivative (or the derivative of the first derivative) gives you the rate of change of the slope of the tangent line with x. As an example, if the second derivative is zero, then the slope does not change with x, and the curve must be a straight line. For this reason, the second derivative is also associated with the *curvature* of a function. Positive second derivatives indicate a tangent line slope that increases with x. Negative second derivatives indicate a tangent line slope that decreases with x.

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Key Takeaways

Below is a centered finite difference approximation for the second derivative at point x_2 where we assume that $x_3 - x_2 = x_2 - x_1 = \Delta x$

$$\left.\frac{\partial^2 p}{\partial x^2}\right|_{x_2} \approx \frac{p(x_3) + p(x_1) - 2p(x_2)}{(\Delta x)^2}$$

Where does this formula come from? We want to approximate how the slope of a tangent line changes in x. We use a centered difference first derivative to estimate the tangent line slope at point half way between x_1 and x_2, which is the point $x_2 - \frac{1}{2}\Delta x$. And then we do the same for points 2 and 3, which gives an estimate for the point $x_2 + \frac{1}{2}\Delta x$. Then we take the change in these two slopes divided by the change in x

(which is also Δx since the points are at the half-way marks).

$$\left.\frac{\partial^2 p}{\partial x^2}\right|_{x_2} \approx \frac{\frac{p(x_3)-p(x_2)}{\Delta x} - \frac{p(x_2)-p(x_1)}{\Delta x}}{\Delta x}$$

Add some algebra to turn it into the previous expression!

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- PressureVsX by Susan Hautala © CC BY-SA (Attribution ShareAlike)

PART II

ADVECTION AND DIFFUSION

This part is under construction, and for a while will just consist of fragments not easily available in the course textbook.

10. The Turbulent Diffusive Flux

We will not go into the details of the physics of turbulence in this course. At this point, our main concern is that turbulence is associated with a much more efficient "diffusion" of properties in a fluid compared to molecular diffusion.

Both larger scale flow and the turbulence are advective processes. The advective flux is the property concentration (C) times the velocity. Here we are going to just look at the flux in the x-direction.

$$F^x_{Advective} = Cu = \left(\overline{C} + C'\right)\left(\overline{u} + u'\right)$$

In the last expression, we have broken both the concentration and the velocity up into two components: a time-mean component indicated by an overbar to indicate that operation, and a component that can fluctuate in time about this mean (but that has zero mean itself). Realistically, this overbar might represent a half hour average – that's about the minimum amount of time it takes to get a good picture of the statistics of the turbulence in the ocean – so the mean flow is not a truly a long-term mean, just a larger scale flow field that is changing more slowly than a timescale of an hour or so.

We will now multiply out this last expression

$$F^x_{Advective} = \overline{C}\overline{u} + \overline{C}u' + \overline{u}C' + u'C'$$

Now we will take the time average of each of these terms

$$F^x_{Advective} = \overline{\overline{C}\overline{u}} + \overline{\overline{C}u'} + \overline{\overline{u}C'} + \overline{u'C'}$$

(1) For the first term, the time average of the product of the two time averages is redundant since these are just constants in time. Physically this represents the advective flux of the property by the larger scale flow field.

(2) The second term is identically zero. The time-average

concentration is just a number and the fluctuating part has zero time average, so their product is zero.

(3) Similarly, the third term is identically zero.

(4) The fourth term represents a time averaged correlation between velocity and property fluctuations. Physically, this term represents the turbulent transport of the property.

Simplifying:

$$F^x_{Advective} = \overline{C}\overline{u} + \overline{u'C'} = \overline{C}\overline{u} + F^x_{Turb}$$

If we had an instrument capable of resolving the turbulent fluctuations we can measure the turbulent transport directly. But these measurements are typically expensive. So we try to model the turbulent transport in terms of things we already know about the larger scale flow and property fields.

One simple and widely used model of this process is to model turbulent transport using Fick's Law of diffusion, but with a diffusivity that is much (several orders of magnitude in the ocean) larger than the molecular diffusivity.

$$F^x_{Turb} = \overline{u'C'} = -K_{Turb}\frac{\partial \overline{C}}{\partial x}$$

We usually drop the overbars from the time-mean quantities (since that is what we are typically trying to model).

Here is an example (artificial) time series of vertical velocity (w) and temperature (T) with zero mean flow, but an underlying upward turbulent transport of heat. The bottom left panel shows the individual w'T' pair at the instant indicated by the red dot in the time series on the top row. The value of the time-averaged turbulent temperature transport is:

$$\overline{w'T'} = 0.278$$

To get the possibly more useful quantity of turbulent heat transport, you would multiply by fluid density and heat capacity.

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PART III

STEADY-STATE MOMENTUM BALANCE(S)

This part is under construction, and for a while will just consist of fragments not easily available in the course textbook.

11. Bernoulli Flow

Bernoulli flow is a term used to describe a velocity field that to first order obeys Bernoulli's Equation. The pressure (P), speed (U) and elevation (*z*) at two points along the same streamline (we will talk more about "streamlines" in class) in Bernoulli flow are related by this equation:

Key Takeaways

The Bernoulli Equation for two points (subscripts 1 and 2) along a streamline in frictionless, incompressible flow:

$$\frac{1}{2}{U_1}^2 + \frac{p_1}{\rho_0} + gz_1 = \frac{1}{2}{U_2}^2 + \frac{p_2}{\rho_0} + gz_2$$

Let's start with a simple case – the one-dimensional eastward component equation (so that U=*u*) for steady (no time-dependence) and frictionless flow:

$$u\frac{\partial u}{\partial x} = PGF = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

where PGF = Pressure Gradient Force. In this form, if we remember that the term on the left is equal to the Lagrangian derivative for this simple 1D flow, we learn that the Lagrangian acceleration is quantitatively related to the PGF via Newton's Law for $a = F/m$. Although the flow is steady, there are spatial changes in velocity (accelerations following a water parcel) – i.e.

Lagrangian accelerations for a moving flow. We are now going to change the left side via a reverse chain-rule for differentiation:

$$\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) \approx \frac{\Delta KE}{\Delta x} = PGF$$

In this form, we can see that the left side is telling us about the spatial gradient of kinetic energy ($KE = \frac{1}{2}u^2$) – the PGF is increasing (if positive) or decreasing (if negative) the kinetic energy to the east along a streamline. For two points separated by a distance Δx, their difference in kinetic energy ΔKE can be written as:

$$\Delta KE = \Delta\left(\frac{1}{2}u^2\right) = PGF * \Delta x$$

The right side is the force times the distance, or the net work done on the system by the PGF which is needed to change the kinetic energy. Thus Bernoulli's Equation is really a statement of conservation of energy in the absence of friction.

Now we will work on the full expression for the PGF:

$$\Delta KE = PGF * \Delta x \; = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} * \Delta x = -\frac{1}{\rho_0}\Delta p$$

and if we write differences using values at point 2 minus point 1:

$$\frac{1}{2}{u_2}^2 - \frac{1}{2}{u_1}^2 = -\frac{1}{\rho_0}(p_2 - p_1)$$

or

$$\frac{1}{2}{u_1}^2 + \frac{p_1}{\rho_0} = \frac{1}{2}{u_2}^2 + \frac{p_2}{\rho_0}$$

which is starting to look like part of the equation in the box.

Of course, it is not actually possible to have a change of just one component of velocity in space because of the continuity equation, so this simple example has to be modified to include at least one more component of velocity. If one of those components is vertical, then we have to take into account changes in potential energy (PE = *gz*). The equation in the box is a general form of the Bernoulli Equation (we will not derive it) that is equivalent to:

$$\Delta KE + \Delta PE = PGF * \Delta x$$

where the KE is now related to the total flow speed, ($KE = \frac{1}{2}U^2$). If you want more in the way of explanation and derivation of the term involving potential energy, see this Khan Academy lesson "What is Bernoulli's Equation?"

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